

MATHEMATICS

A LEAST SQUARES INEQUALITY FOR MAXIMUM LIKELIHOOD
ESTIMATES OF ORDERED PARAMETERS ¹⁾

BY

CONSTANCE VAN EEDEN

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1. *Introduction*

In this paper the results of a further investigation on the maximum likelihood estimates of partially or completely ordered parameters will be given. One of these results is a generalization of the following inequality for the binomial case, which may be found in [1] (p. 644).

If ²⁾

$$(1.1) \quad P[\mathbf{x}_i = 1] = \theta_i, \quad P[\mathbf{x}_i = 0] = 1 - \theta_i \quad (i = 1, \dots, k)$$

and

$$(1.2) \quad a_i \stackrel{\text{def}}{=} \sum_{\gamma=1}^{n_i} x_{i,\gamma}, \quad b_i \stackrel{\text{def}}{=} n_i - a_i \quad (i = 1, \dots, k),$$

then

$$(1.3) \quad \sum_{i=1}^k n_i \left(y_i - \frac{a_i}{n_i} \right)^2 \geq \sum_{i=1}^k n_i \left\{ \left(t_i - \frac{a_i}{n_i} \right)^2 + (t_i - y_i)^2 \right\}$$

for each point $(y_1, \dots, y_k) \in D$.

The inequality (1.3) is equivalent with

$$(1.4) \quad \sum_{i=1}^k n_i (t_i - y_i) \left(t_i - \frac{a_i}{n_i} \right) \leq 0 \quad \text{for each point } (y_1, \dots, y_k) \in D.$$

In this paper the inequality (1.4) will be generalized for the case of partially or completely ordered parameters of other probability distributions. The problem will be treated in section 2 and in section 3 some examples will be given.

2. *The problem*

In this paper we suppose that, for each subset M of E , there exists a pair of values (i, j) with

$$(2.1) \quad \begin{cases} 1. & i \in M, j \in \bar{M}, \\ 2. & \alpha_{i,j} \neq 0. \end{cases}$$

¹⁾ Report SP 60 of the Statistical Department of the Mathematical Centre, Amsterdam.

²⁾ The notation in this paper is the same as the one used in [2] and [3].

This may be supposed without any loss of generality for if there exists a subset M of E not satisfying this condition then (cf. theorem IV in [2]) the estimates $t_1, \dots, t_k$ may be found by separately maximizing $\sum_{i \in M} L_i(y_i)$ in the domain

$$(2.2) \quad D_1: \begin{cases} \alpha_{i,j}(y_i - y_j) \leq 0 \\ y_i \in I_i \end{cases} \quad (i, j \in M)$$

and $\sum_{i \in \bar{M}} L_i(y_i)$ in the domain

$$(2.3) \quad D_2: \begin{cases} \alpha_{i,j}(y_i - y_j) \leq 0 \\ y_i \in I_i \end{cases} \quad (i, j \in \bar{M})^3).$$

Let J_i be the set of all values of y for which $F_i(x_i | y)$ is a distribution function ($i=1, \dots, k$); we suppose J_i to be an interval. Let further for any subset M of E

$$(2.4) \quad J_M \stackrel{\text{def}}{=} \bigcap_{i \in M} J_i.$$

In this paper we suppose that the following condition is satisfied.

(2.5) Condition: For each M with $J_M \neq \emptyset$ the function $L_M(z)$ is strictly unimodal in J_M .⁴⁾

Let w_M be the value of z which maximizes $L_M(z)$ in J_M and let w_i denote the value of y which maximizes $L_i(y)$ in J_i ($i=1, \dots, k$). Then if I_i is the interval (c_i, d_i) and if (cf. [3], section 2) v_M is the value of z which maximizes $L_M(z)$ in I_M

$$(2.6) \quad \begin{cases} v_M = w_M & \text{if } \max_{i \in M} c_i \leq w_M \leq \min_{i \in M} d_i, \\ v_M = \max_{i \in M} c_i & \text{if } w_M < \max_{i \in M} c_i, \\ v_M = \min_{i \in M} d_i & \text{if } w_M > \min_{i \in M} d_i. \end{cases}$$

Now let E_0 be a subset of E with

$$(2.7) \quad \begin{cases} 1. & t_i \neq v_i & \text{for each } i \in E_0, \\ 2. & t_i = v_i & \text{for each } i \in \bar{E}_0, \end{cases}$$

then

Lemma I: The estimates $t_1, \dots, t_k$ may also be found by separately maximizing $\sum_{i \in E_0} L_i(y_i)$ in the domain

$$(2.8) \quad D': \begin{cases} \alpha_{i,j}(y_i - y_j) \leq 0 \\ y_i \in I_i \end{cases} \quad (i, j \in E_0)$$

³⁾ In the definitions of the domains D_1 , D_2 , D' , D'' and D'_v (cf. (2.8), (2.9) and (2.21)) the coordinates which are not mentioned may assume any values.

⁴⁾ If $J_i = I_i$ for each $i \in E$ then this condition is identical with condition (4.3) in [2].

and $\sum_{i \in \bar{E}_0} L_i(y_i)$ in the domain

$$(2.9) \quad D'': y_i \in I_i \quad (i \in \bar{E}_0).^5$$

Proof:

The function $\sum_{i \in \bar{E}_0} L_i(y_i)$ attains its maximum in D'' for $y_i = v_i = t_i$ ($i \in \bar{E}_0$).

Further the function $L(y_1, \dots, y_k)$ attains its maximum under the conditions $y_i = t_i$ ($i \in \bar{E}_0$) in D for $y_i = t_i$ ($i = 1, \dots, k$), i.e. the function $\sum_{i \in \bar{E}_0} L_i(t_i) + \sum_{i \in E_0} L_i(y_i)$ attains its maximum in D for $y_i = t_i$ ($i = 1, \dots, k$). Thus $\sum_{i \in E_0} L_i(y_i)$ attains its maximum in D' for $y_i = t_i$ ($i \in E_0$).

Now let M_0 be a subset of E_0 with

$$(2.10) \quad t_i = t_j \text{ for each pair of values } (i, j) \in M_0.$$

Theorem I: If $\delta_1, \dots, \delta_k$ satisfy

$$(2.11) \quad \begin{cases} 1. & \sum_{i \in M_0} \delta_i(w_{M_0} - w_i) = 0 \text{ for each } M_0 \subset E_0 \text{ satisfying (2.10),} \\ 2. & \delta_i > 0 \text{ for each } i \in E, \end{cases}$$

then the function

$$(2.12) \quad Q = Q(y_1, \dots, y_k) \stackrel{\text{def}}{=} \sum_{i=1}^k \delta_i (y_i - w_i)^2$$

attains its minimum in D for $y_i = t_i$ ($i = 1, \dots, k$).

Proof:

If M is a subset of E then $-\sum_{i \in M} \delta_i(z - w_i)^2$ is a strictly unimodal function of z ; thus, analogous to theorem I in [2], $-Q$ possesses a unique maximum in D , i.e. Q possesses a unique minimum in D .

Every term of the function $\sum_{i \in \bar{E}_0} \delta_i(y_i - w_i)^2$ (and thus the function itself) attains its minimum in D'' (cf. (2.9)) for $y_i = v_i = t_i$ ($i \in \bar{E}_0$). Lemma I then implies that it is sufficient to prove that the minimum of

$$\sum_{i \in E_0} \delta_i(y_i - w_i)^2$$

in D' (cf. (2.8)) coincides with the maximum of $\sum_{i \in E_0} L_i(y_i)$ in D' .

We first prove this (by induction⁶) under the following stronger condition for $\delta_1, \dots, \delta_k$

$$(2.13) \quad \begin{cases} 1. & \sum_{i \in M} \delta_i(w_M - w_i) = 0 \text{ for each } M \subset E_0, \\ 2. & \delta_i > 0 \text{ for each } i \in E_0. \end{cases}$$

⁵) Cf. footnote 3.

⁶) Cf. the proof of theorem I in [2], which runs along the same ways.

Let $M_\nu (\nu=1, \dots, N)$ be subsets of E_0 with

$$(2.14) \quad \left\{ \begin{array}{l} 1. \quad \bigcup_{\nu=1}^N M_\nu = E_0, \\ 2. \quad M_{\nu_1} \cap M_{\nu_2} = 0 \text{ for each pair of values } (\nu_1, \nu_2) \text{ with } \nu_1 \neq \nu_2, \\ 3. \quad I_{M_\nu} \neq 0 \text{ for each } \nu=1, \dots, N. \end{array} \right.$$

Let further (cf. (4.6) in [2])

$$(2.15) \quad G_N \stackrel{\text{def}}{=} \prod_{\nu=1}^N I_{M_\nu}$$

and (cf. (4.7) and (4.8) in [2])

$$(2.16) \quad \left\{ \begin{array}{l} Q_{M_\nu}(z) \stackrel{\text{def}}{=} \sum_{i \in M_\nu} \delta_i (z - w_i)^2, \\ Q'(z_1, \dots, z_N) \stackrel{\text{def}}{=} \sum_{\nu=1}^N Q_{M_\nu}(z_\nu) \end{array} \right.$$

and (cf. (4.9) in [2])

$$(2.17) \quad D'_{N,s} \stackrel{\text{def}}{=} D' \cap G_N,$$

where s denotes the number of essential restrictions defining D' . Then the function $Q_{M_\nu}(z)$ attains its minimum in the interval $(-\infty, +\infty)$ for

$$(2.18) \quad z = \frac{\sum_{i \in M_\nu} \delta_i w_i}{\sum_{i \in M_\nu} \delta_i} = w_{M_\nu} \quad (\text{cf. (2.13)})$$

and the fact that $-Q_{M_\nu}(z)$ is strictly unimodal in the interval $(-\infty, +\infty)$ then entails that $Q_{M_\nu}(z)$ attains its minimum in I_{M_ν} for $z = v_M$ (cf. (2.6)). The minimum of Q' in G_N thus coincides with the maximum of L' in $G_N = D'_{N,0}$.

Now suppose that it has been proved that the minimum of Q' in $D'_{N,s}$ coincides with the maximum of L' in $D'_{N,s}$ for each $s \leq s_0$, for each partition $M_1, \dots, M_N$ of E_0 satisfying (2.14) and for each N .

We then prove that the same holds for s_0+1 essential restrictions. Consider, for a given partition $M_1, \dots, M_N$ satisfying (2.14) a domain D'_{N,s_0+1} and the domain D'_{N,s_0} which is obtained by omitting one of the essential restrictions defining D'_{N,s_0+1} . Let this be the restriction $z_{\nu_1} \leq z_{\nu_2}$. Then $D'_{N,s_0+1} \subset D'_{N,s_0}$. The minimum of Q' in D'_{N,s_0} coincides with the maximum of L' in D'_{N,s_0} in (say) the point $(z_1^0, \dots, z_N^0)$ and the following two cases may be distinguished:

1. $z_{\nu_1}^0 \leq z_{\nu_2}^0$; then $(z_1^0, \dots, z_N^0) \in D'_{N,s_0+1}$. Thus in this case the minimum of Q' in D'_{N,s_0+1} coincides with the maximum of L' in D'_{N,s_0+1} .

2. $z_{\nu_1}^0 > z_{\nu_2}^0$; then (cf. theorem II in [2]) Q' attains its minimum (and L' its maximum) in D'_{N,s_0+1} for $z_{\nu_1} = z_{\nu_2}$. The domain D'_{N,s_0+1} reduces, with $z_{\nu_1} = z_{\nu_2}$, to a domain $D'_{N-1,s_0'}$ with $s_0' \leq s_0$ and the minimum of Q' under the condition $z_{\nu_1} = z_{\nu_2}$ in $D'_{N-1,s_0'}$ coincides with the maximum of L' under

the condition $z_{v_1} = z_{v_2}$ in $D'_{N-1, s_0'}$. Thus if $\delta_i (i \in E_0)$ satisfy (2.13) then the minimum of Q' in $D'_{N, s}$ coincides with the maximum of L' in $D'_{N, s}$. This holds for each N , i.e. it holds for $N = k'$, if k' is the number of elements of E_0 . Thus if $\delta_i (i \in E_0)$ satisfy (2.13) then the minimum of $\sum_{i \in E_0} \delta_i (y_i - w_i)^2$ in $D'_{k', s} = D'$ coincides with the maximum of $\sum_{i \in E_0} L_i(y_i)$ in D' .

We now prove the theorem under condition (2.11). Let $E_v (v = 1, \dots, K)$ be subsets of E_0 with

$$(2.19) \quad \left\{ \begin{array}{l} 1. \quad \bigcup_{v=1}^K E_v = E_0, \\ 2. \quad t_i < t_j \text{ for each pair of values } (i, j) \text{ with } i \in E_{v_1}, \\ \quad \quad \quad j \in E_{v_2} (v_1 < v_2; v_1, v_2 = 1, \dots, K), \\ 3. \quad t_i = t_j \text{ for each pair of values } (i, j) \in E_v (v = 1, \dots, K), \end{array} \right.$$

then (2.11) is identical with

$$(2.20) \quad \left\{ \begin{array}{l} 1. \quad \sum_{i \in M} \delta_i (w_M - w_i) = 0 \quad \text{for each } M \subset E_v (v = 1, \dots, K), \\ 2. \quad \delta_i > 0 \quad \text{for each } i \in E. \end{array} \right.$$

Further it follows from lemma II in [3] and theorem IV in [2] that the maximum of $\sum_{i \in E_0} L_i(y_i)$ in D' may also be found by maximizing, for $v = 1, \dots, K$, $\sum_{i \in E_v} L_i(y_i)$ in the domain ⁷⁾

$$(2.21) \quad D'_v: \begin{cases} \alpha_{i,j} (y_i - y_j) \leq 0 \\ y_i \in I_i \end{cases} \quad (i, j \in E_v).$$

Further the fact that $\delta_1, \dots, \delta_k$ satisfy (2.20) entails that the minimum of $\sum_{i \in E_v} \delta_i (y_i - w_i)^2$ in D'_v coincides with the maximum of $\sum_{i \in E_v} L_i(y_i)$ in $D'_v (v = 1, \dots, K)$. This proves the theorem under condition (2.11).

Remark 1:

In the proof of theorem I the fact has been used that the maximum of L in D coincides with the maximum of L in the domain

$$(2.22) \quad B \stackrel{\text{def}}{=} D'' \cap \bigcap_{v=1}^K D'_v. \text{ } ^8)$$

The same holds for the minimum of Q .

Theorem II: If $\delta_1, \dots, \delta_k$ satisfy (2.11) and if $t_i \neq w_i$ for at least one value of $i \in E$ then the ellipsoid

$$(2.23) \quad \sum_{i=1}^k \delta_i \left(y_i - \frac{t_i + w_i}{2} \right)^2 = \sum_{i=1}^k \delta_i \left(\frac{t_i - w_i}{2} \right)^2$$

touches the domains B (and D) in the point $(t_1, \dots, t_k)$.

⁷⁾ Cf. footnote 3.

⁸⁾ The domain D is independent of $t_1, \dots, t_k$, but B depends on these estimates.

Proof:

If $t_i \neq w_i$ for at least one value of $i \in E$ then $(t_1, \dots, t_k)$ is a borderpoint of B and $(w_1, \dots, w_k) \notin B$. Further, if $0 < \beta \leq 1$, then

$$\{\beta w_1 + (1-\beta)t_1, \dots, \beta w_k + (1-\beta)t_k\} \notin B.$$

This may be seen as follows. From lemma I in [2] it follows that

$$\sum_{i=1}^k L_i \{\beta w_i + (1-\beta)t_i\}$$

is a monotone increasing function of β in the interval $0 \leq \beta \leq 1$. Thus if $0 < \beta \leq 1$ then

$$\sum_{i=1}^k L_i \{\beta w_i + (1-\beta)t_i\} > \sum_{i=1}^k L_i(t_i).$$

The fact that L attains its maximum in B in the point $(t_1, \dots, t_k)$ then implies $\{\beta w_1 + (1-\beta)t_1, \dots, \beta w_k + (1-\beta)t_k\} \notin B$.

Now let

$$(2.24) \quad \begin{cases} y_i \stackrel{\text{def}}{=} \sqrt{\delta_i} \left(y_i - \frac{t_i + w_i}{2} \right), \\ w_i \stackrel{\text{def}}{=} \sqrt{\delta_i} \left(w_i - \frac{t_i + w_i}{2} \right) = \sqrt{\delta_i} \frac{w_i - t_i}{2}, \\ t_i \stackrel{\text{def}}{=} \sqrt{\delta_i} \left(t_i - \frac{t_i + w_i}{2} \right) = -\sqrt{\delta_i} \frac{w_i - t_i}{2}, \end{cases} \quad (i = 1, \dots, k)$$

then (2.23) reduces to

$$(2.25) \quad \sum_{i=1}^k y_i'^2 = \sum_{i=1}^k t_i'^2 \left(= \sum_{i=1}^k w_i'^2 \right)$$

and B reduces to a domain B' . Further $(t'_1, \dots, t'_k)$ is a borderpoint of B' , $(w_1, \dots, w_k) \notin B'$ and, for each β with $0 < \beta \leq 1$,

$$\{\beta w'_1 + (1-\beta)t'_1, \dots, \beta w'_k + (1-\beta)t'_k\} \notin B'.$$

From (2.24) follows

$$(2.26) \quad \sum_{i=1}^k \delta_i (y_i - w_i)^2 = \sum_{i=1}^k (y'_i - w'_i)^2.$$

From theorem I and remark 1 then follows that $\sum_{i=1}^k (y'_i - w'_i)^2$ attains its minimum in B' in the point $(t'_1, \dots, t'_k)$, thus the sphere (2.25) touches B' in $(t'_1, \dots, t'_k)$; i.e. the ellipsoid (2.23) touches B in $(t_1, \dots, t_k)$.

We now prove the following lemma:

Lemma II: Let C be a convex domain and S a point on its boundary. Let K_s be an ellipsoid touching C on the outside in S and let the diameter of K_s , passing through S , intersect K_s in a point U . Let further Y be a point inside C or on its boundary and K_Y an ellipsoid with diameter YU , with axes parallel to those of K_s and with the length of the axes proportional to those of K_s . Then S lies inside or on K_Y .

Proof:

We apply a linear transformation such that K_s reduces to a sphere K'_s ; then K_Y reduces to a sphere K'_Y , C to a convex domain C' , S to a point S' on the boundary of C' and Y to a point Y' inside or on the boundary of C' . The sphere K'_s touches C' in S' and it may easily be seen that S' lies inside or on K'_Y .

Theorem III: If $\delta_1, \dots, \delta_k$ satisfy (2.11) then

$$(2.27) \quad \sum_{i=1}^k \delta_i (t_i - w_i)(t_i - Y_i) \leq 0 \text{ for each point } (Y_1, \dots, Y_k) \in B.$$

Proof:

If $t_i = w_i$ for each $i \in E$ then (2.11) reduces to

$$(2.28) \quad \delta_i > 0 \text{ for each } i \in E.$$

Then the theorem is immediately clear.

If $t_i \neq w_i$ for at least one value of $i \in E$ then (cf theorem II) the ellipsoid (2.23) touches B in the point $(t_1, \dots, t_k)$. Thus if $(Y_1, \dots, Y_k)$ is a point in B then it follows from lemma II that $(t_1, \dots, t_k)$ lies inside or on the ellipsoid

$$(2.29) \quad \sum_{i=1}^k \delta_i \left(y_i - \frac{w_i + Y_i}{2} \right)^2 = \sum_{i=1}^k \delta_i \left(\frac{w_i - Y_i}{2} \right)^2,$$

i.e. $t_1, \dots, t_k$ satisfy

$$(2.30) \quad \sum_{i=1}^k \delta_i \left(t_i - \frac{w_i + Y_i}{2} \right)^2 \leq \sum_{i=1}^k \delta_i \left(\frac{w_i - Y_i}{2} \right)^2$$

and (2.30) is identical with

$$(2.31) \quad \sum_{i=1}^k \delta_i (t_i - w_i)(t_i - Y_i) \leq 0.$$

Further it follows from the foregoing that the following theorem holds.

Theorem IV: If $\delta_1, \dots, \delta_k$ satisfy (2.11) then there exists exactly one point $(y_1, \dots, y_k) \in B$ satisfying the inequalities

$$(2.32) \quad \sum_{i=1}^k \delta_i (y_i - w_i)(y_i - Y_i) \leq 0 \quad (Y_1, \dots, Y_k) \in B.$$

Thus if $\delta_1, \dots, \delta_k$ satisfy (2.11) and are independent of $t_1, \dots, t_k$ then the estimates $t_1, \dots, t_k$ may also be found by minimizing $Q(y_1, \dots, y_k)$ in D or by solving the inequalities (2.32) with $(Y_1, \dots, Y_k) \in D$.

3. Examples

If (cf. section 1)

$$(3.1) \quad P[\mathbf{x}_i = 1] = \theta_i, \quad P[\mathbf{x}_i = 0] = 1 - \theta_i \quad (i = 1, \dots, k)$$

and

$$(3.2) \quad a_i \stackrel{\text{def}}{=} \sum_{\gamma=1}^{n_i} x_{i,\gamma}, \quad b_i \stackrel{\text{def}}{=} n_i - a_i \quad (i = 1, \dots, k),$$

then

$$(3.3) \quad w_M = \frac{\sum_{i \in M} a_i}{\sum_{i \in M} n_i} = \frac{\sum_{i \in M} n_i w_i}{\sum_{i \in M} n_i}.$$

Thus if $\delta_i = n_i$ ($i = 1, \dots, k$), then $\delta_1, \dots, \delta_k$ satisfy (2.11) and are independent of $w_1, \dots, w_k$; i.e. the estimates $t_1, \dots, t_k$ may also be found by minimizing

$$(3.4) \quad Q(y_1, \dots, y_k) = \sum_{i=1}^k n_i (y_i - w_i)^2$$

in D and $t_1, \dots, t_k$ satisfy

$$(3.5) \quad \sum_{i=1}^k n_i (t_i - w_i) (t_i - Y_i) \leq 0 \quad \text{for each point } (Y_1, \dots, Y_k) \in B.$$

If $\mathbf{x}_i$ possesses a normal distribution with mean θ_i and variance σ_i^2 ($i = 1, \dots, k$), where σ_i^2/σ_j^2 is known for each pair of values (i, j) then

$$(3.6) \quad w_M = \frac{\sum_{i \in M} \frac{1}{\sigma_i^2} \sum_{\gamma=1}^{n_i} x_{i,\gamma}}{\sum_{i \in M} \frac{n_i}{\sigma_i^2}} = \frac{\sum_{i \in M} \frac{n_i w_i}{\sigma_i^2}}{\sum_{i \in M} \frac{n_i}{\sigma_i^2}},$$

thus $\delta_i = n_i/\sigma_i^2$ ($i = 1, \dots, k$) satisfies (2.11); i.e. the estimates $t_1, \dots, t_k$ may also be found by minimizing ⁹⁾

$$(3.7) \quad Q(y_1, \dots, y_k) = \sum_{i=1}^k \frac{n_i}{\sigma_i^2} (y_i - w_i)^2$$

in D and $t_1, \dots, t_k$ satisfy the inequalities

$$(3.8) \quad \sum_{i=1}^k \frac{n_i}{\sigma_i^2} (t_i - w_i) (t_i - Y_i) \leq 0 \quad \text{for each point } (Y_1, \dots, Y_k) \in B.$$

In the same way it may be proved that $\delta_i = n_i$ ($i = 1, \dots, k$) satisfies (2.11) if

1. $\mathbf{x}_i$ possesses a normal distribution with known mean μ_i and variance θ_i ($i = 1, \dots, k$),
2. $\mathbf{x}_i$ possesses an exponential distribution

$$(3.9) \quad P[\mathbf{x}_i \leq x] = 1 - e^{-\frac{x}{\theta_i}} \quad (i = 1, \dots, k).$$

In all these cases the estimates $t_1, \dots, t_k$ are the ordinary least squares estimates in D .

If on the other hand $\mathbf{x}_i$ possesses a rectangular distribution "between"

⁹⁾ This also follows from

$$\begin{aligned} L(y_1, \dots, y_k) &= -\frac{1}{2} \sum_{i=1}^k n_i \ln 2\pi\sigma_i^2 - \frac{1}{2} \sum_{i=1}^k \frac{\sum_{\gamma=1}^{n_i} (x_{i,\gamma} - y_i)^2}{\sigma_i^2} = \\ &= -\frac{1}{2} \sum_{i=1}^k n_i \ln 2\pi\sigma_i^2 - \frac{1}{2} \sum_{i=1}^k \frac{\sum_{\gamma=1}^{n_i} (x_{i,\gamma} - w_i)^2}{\sigma_i^2} - \frac{1}{2} \sum_{i=1}^k \frac{n_i}{\sigma_i^2} (y_i - w_i)^2. \end{aligned}$$

0 and $\theta_i (i=1, \dots, k)$ then

$$(3.10) \quad w_M = \max_{i \in M} \max_{1 \leq \gamma \leq n_i} x_{i,\gamma} = \max_{i \in M} w_i.$$

Thus in this case there are no numbers $\delta_1, \dots, \delta_k$ satisfying (2.11).

Note added in proof

If v'_M is the value of z which maximizes $Q_M(z)$ in I_M then the theorems I–IV also hold if $\delta_1, \dots, \delta_k$ are chosen in such a way that

$$\left\{ \begin{array}{l} 1. \max_T v'_{T \cap E_\nu} = \min_S v'_{S \cap E_\nu} = v_{E_\nu} \quad (\nu = 1, \dots, K), \\ 2. \delta_i > 0 \text{ for each } i \in E. \end{array} \right.$$

The proof, which is based on formula (2.5) in [3] will be given in a following paper.

Mathematical Centre, Amsterdam

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